Homework #3 of Topology II
Due Date: Feb 19, 2018

1. If $m < p$, show that every smooth map $M^m \to S^p$ is homotopic to a constant.

2. (a) Compute the degree of the antipodal map, $S^k \to S^k$, $x \to -x$.
   (b)Prove that the antipodal map is homotopic to the identity if and only if k is odd.
   (c) Show that there exists a nonvanishing vector field on S^k if and only if k is odd.

3. Suppose that $W \xrightarrow{\text{g}} X \xrightarrow{\text{f}} Y$ is a sequence of smooth maps with $dim(W) = dim(X) = dim(Y)$. Prove that
$$deg(f \circ g) = deg(f)deg(g).$$

4. **Definition:** Let X be a compact manifold, Z a closed smooth manifold of Y. Suppose X, Y, Z are boundaryless and suppose that $dim(X) + dim(Z) = dim(Y)$. Let $f : X \to Y$ be a smooth map transversing to Z. Define the mod 2 intersection number of f with Z to be the number of the points in $f^{-1}(Z)$ mod 2, denoted by $I_2(f, Z)$.

   Show that if two smooth map $f_1, f_2 : X \to Y$ transversal to Z are homotopic, then $I_2(f_1, Z) = I_2(f_2, Z)$.
   (Hint:you may need the following Extension Theorem: Let Z be a closed submanifold of Y, both boundaryless. Suppose W is a compact manifold and $C \subset W$ is closed subset. If $f \colon W \to Y$ is transversal to Z on C and ∂f is transversal to Z on $\partial W \cap C$, then there exists a smooth map g homotopic to f such that $g \pitchfork Z$ and $\partial g \pitchfork Z$, and on a neighborhood of C we have $f = g$.).

5. Given disjoint manifolds $M, N \subset \mathbb{R}^{k+1}$, the linking map
$$\lambda : M \times N \to S^k$$
   is defined by $\lambda(x, y) = (x-y)/\|x - y\|$. If M and N are compact, oriented and boundaryless, with total dimension $m + n = k$, then the degree λ is called the linking number $l(M, N)$. Prove that
$$l(M, N) = (-1)^{(m+1)(n+1)} l(N, M).$$
   If M is the boundary of an oriented manifold $X \subset \mathbb{R}^{k+1}$ disjoint from N, prove that $l(M, N) = 0$.

   Define the linking number of disjoint manifolds in the sphere S^{m+n+1}.

6. If $y \neq z$ are regular values for the smooth map $f : S^{2p-1} \to S^p$, then the manifolds $f^{-1}(y)$ and $f^{-1}(z)$ can be naturally oriented, hence the linking number $l(f^{-1}(y), f^{-1}(z))$ is defined.

   (a) Show that the linking number is locally constant as a function of y.
   (b) If y and z are regular values of g also, where

   $$\|f(x) - g(x)\| < \|y - z\|$$

   for all x, prove that

   $$l(f^{-1}(y), f^{-1}(z)) = l(g^{-1}(y), f^{-1}(z)) = l(g^{-1}(y), g^{-1}(z)).$$

   (Hint: Prove that $\|f(x) - g(x)\| < \|y - z\|$ implies that $g^{-1}(y)$ is disjoint from $f^{-1}(z)$ and the homotopy

   $$f_t(x) = \frac{tf(x) + (1-t)g(x)}{\|tf(x) + (1-t)g(x)\|}$$

   makes sense.

   (c) Prove that $l(f^{-1}(y), f^{-1}(z))$ depends only on the homotopy class of f, and does not depends on the choice of y and z.